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HAUSDORFF MEASURE FUNCTIONS IN THE SPACE OF COMPACT SUBSETS OF THE UNIT INTERVAL

BY

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ABSTRACT. The work done in this paper is the result of an attempt to classify those functions h for which the corresponding Hausdorff measure of $\mathfrak{F}[0, 1]$ is zero. A partial characterization is achieved and in doing this some problems of E. Boardman are solved.

Introduction. If C is a compact nonempty subset of R^1 then $\mathfrak{F}[C]$ denotes the compact metric space consisting of all nonempty compact subsets C endowed with the Hausdorff metric. If h is a continuous, increasing function defined on the nonnegative real numbers with $h(0) = 0$ we shall denote by $h - m$ the Hausdorff measure generated by h (see [3]). In [1] it is shown that for each function h either $h - m(\mathfrak{F}[0, 1]) = 0$ or $\mathfrak{F}[0, 1]$ has non-σ-finite h-measure. For $\alpha > 0$, the functions h_α, g_α are defined by

$$h_\alpha(t) = 2^{-\alpha t^{-1}} \quad \text{and} \quad g_\alpha(t) = 2^{-t^{-\alpha}} \quad \text{for } t > 0$$

and $h_\alpha(0) = g_\alpha(0) = 0$. Then Boardman shows that $g_\alpha - m(\mathfrak{F}[0, 1]) = \infty$ for all $0 < \alpha < 1$ and that $h_1 - m(\mathfrak{F}[0, 1]) = 0$. The evaluation of

$$h_\alpha - m(\mathfrak{F}[0, 1]) \quad \text{for } 0 < \alpha < 1$$

is left as an open problem.

The work done in this paper is the result of an attempt to classify those functions h for which $h - m(\mathfrak{F}[0, 1]) = 0$. The main result is

THEOREM 1. *Let h be such that* $\liminf_{x\to 0}\{-(x \log h(x))^{-1}\} < \infty$, *then* $h - m(\mathfrak{F}[0, 1]) = 0$.

Thus it follows that $h_\alpha - m(\mathfrak{F}[0, 1]) = 0$ whenever $0 < \alpha \leqslant 1$. We observe, from [1], that if there is an $\alpha < 1$ with $\liminf_{x\to 0}\{-(x^\alpha \log h(x))^{-1}\} > 0$, then $h - m(\mathfrak{F}[0, 1]) = \infty$. It is clear, therefore, that this characterization is only

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partial as it gives no information about those functions for which $\lim_{x\to 0}\{-(x \log h(x))^{-1}\} = \infty$, and yet $\liminf_{x\to 0}\{-(x^{\alpha} \log h(x))^{-1}\} = 0$ for all $\alpha < 1$. We shall denote by $\mathcal{K}$ the collection of all such functions. Now if $f_n(x) = 2^{-(nx)^{-1}}$, then, by Theorem 1, $f_n - m(\mathcal{F}[0, 1]) = 0$ for $n = 1, 2, \ldots$. But by a result of Rogers and Taylor [4] there is a function h such that $h - m(\mathcal{F}[0, 1]) = 0$ and $\lim_{x\to 0+} f_n(x)(h(x))^{-1} = 0$ for $n = 1, 2, \ldots$. So there is a function $h \in \mathcal{K}$ for which $h - m(\mathcal{F}[0, 1]) = 0$. Conversely, it follows, by a slight generalization of the methods of [1], that

Theorem 2. *There is a function $h \in \mathcal{K}$ for which $h - m(\mathcal{F}[0, 1]) = \infty$.*

If we combine these results with some work of Dvoretzky [2] we see that if the unit interval has σ-finite h-measure then $2^{-h^{-1}} - m(\mathcal{F}[0, 1]) = 0$. This is interesting in that it relates some properties of [0, 1] to some of $\mathcal{F}[0, 1]$, and thus might suggest working in the more general setting of an arbitrary compact set K in place of [0, 1]. Unfortunately, one of our examples, combined with Dvoretzky's result also shows there is a function h such that the unit interval has non-σ-finite h-measure and yet $2^{-h^{-1}} - m(\mathcal{F}[0, 1]) = 0$.

I should like to thank E. Boardman for providing me with a preprint of her paper [1] and for the interesting correspondence we had concerning this work. I am also indebted to the referee for his constructive criticism of an earlier form of this paper.

Preliminaries. Let C be a compact nonempty subset of R^1 and let $x_i \in C$ for $i = 1, 2, \ldots, n$; then $\{x_1, x_2, \ldots, x_n\} \in \mathcal{F}[C]$. The sphere in $\mathcal{F}[C]$, centre $\{x_1, x_2, \ldots, x_n\}$ and radius r is precisely the set of compact subsets K of C such that $K \cap [x_i - r, x_i + r] \neq \emptyset$ for $i = 1, 2, \ldots, n$. Also if $C \subset \bigcup_{i=1}^{t} I_i$ where the I_i are closed intervals each of length l, then for $K \in \mathcal{F}[C]$, $K \in \mathcal{F}[\bigcup_{i\in A} I_i]$ and $K \cap I_i \neq \emptyset$ for each $i \in A$ where A is some nonempty subset of $\{1, 2, \ldots, t\}$. In fact K belongs to the sphere in $\mathcal{F}[\bigcup_{i=1}^{t} I_i]$, centre $\{x_i\}_{i\in A}$ and radius $\frac{1}{2}l$ where x_i is the midpoint of I_i. Thus $\mathcal{F}[C]$ is contained in a union of $2^t - 1$ spheres each of diameter l.

Proof of Theorem 1.

Lemma 1. *Let $\alpha \in (0, 1)$, $S = \{x_i\}$ be a sequence of positive real numbers with $\lim_{i\to\infty} x_i = 0$ and I_j $(j = 1, 2, \ldots, t)$ be closed intervals such that $\sum_{j=1}^{t} d(I_j) < \alpha$. Then, given any $\varepsilon > 0$, there is a sequence $\{\mathcal{G}_n\}_{n=1}^{N}$ of sets such that $\mathcal{F}[\bigcup_{j=1}^{t} I_j] \subset \bigcup_{n=1}^{N} \mathcal{G}_n$, $\sum_{n=1}^{N} h_\alpha(d(\mathcal{G}_n)) < \varepsilon$ and $d(\mathcal{G}_n) \in S$ for $n = 1, 2, \ldots, N$.*

Proof. Let $\eta > 0$ be such that $\sum_{j=1}^{t} d(I_j) < \alpha - \eta$. Choose i_0 such that, for $i \geqslant i_0$, $2^{t - \eta x_i^{-1}} < \varepsilon$. Now I_j can be covered by $[d(I_j)x_i^{-1}] + 1$ closed intervals each length x_i, where the square brackets denote the integer part function. Thus $\bigcup_{j=1}^{t} I_j$ can be covered by $N(i) = \sum_{j=1}^{t} \{[d(I_j)x_i^{-1}] + 1\}$ closed intervals each of length x_i. Now

$$N(i) \leqslant x_i^{-1} \sum_{j=1}^{t} d(I_j) + t < (\alpha - \eta)x_i^{-1} + t$$

and so

$$(2^{N(i)} - 1)h_\alpha(x_i) < 2^{(\alpha-\eta)x_i^{-1}+t-\alpha x_i^{-1}} < \varepsilon.$$

Denote by $\{\mathcal{G}_n\}$ any enumeration of the sets of compact subsets of the nonempty subcollections of the covering intervals each of length x_i. This completes the proof of the lemma.

LEMMA 2. *Let α and S be as in Lemma 1 and denote by $\mathcal{F}(\alpha)$ those compact sets of Lebesgue measure $< \alpha$. Then, given any $\varepsilon > 0$, there is a sequence $\{\mathcal{G}_n\}$ of sets such that $\mathcal{F}(\alpha) \subset \bigcup_{n=1}^{\infty} \mathcal{G}_n$, $\sum_{n=1}^{\infty} h_\alpha(d(\mathcal{G}_n)) < \varepsilon$ and $d(\mathcal{G}_n) \in S$ for $n = 1, 2, \ldots$.*

PROOF. Let $K \in \mathcal{F}(\alpha)$; then there is a finite covering $J_1, J_2, \ldots, J_t$ of K by open intervals with rational endpoints such that $\sum_{i=1}^{t} d(J_i) < \alpha$. For $i = 1, 2, \ldots, t$ put $I_i = \bar{J}_i$. Then the intervals $I_1, I_2, \ldots, I_t$ satisfy the conditions of Lemma 1 and $K \in \mathcal{F}[\bigcup_{j=1}^{t} I_j]$. Now there are only enumerably many closed intervals with rational endpoints and thus only enumerably many finite unions of such closed intervals. So we may denote by $\mathcal{I}_1, \mathcal{I}_2, \ldots$ those of the finite unions of intervals whose total length is less than α. Then by Lemma 1 there is, for each integer j, a sequence $\{\mathcal{G}_n^j\}_{n=1}^{N(j)}$ of sets such that $\mathcal{F}[\mathcal{I}_j] \subset \bigcup_{n=1}^{N(j)} \mathcal{G}_n^j$; $\sum_{n=1}^{N(j)} h_\alpha(d(\mathcal{G}_j^n)) < \varepsilon 2^{-j}$ and $d(\mathcal{G}_n^j) \in S$ for $n = 1, 2, \ldots, N(j)$. But we have shown that $\mathcal{F}(\alpha) \subset \bigcup_{j=1}^{\infty} \mathcal{F}[\mathcal{I}_j]$ and so

$$\mathcal{F}(\alpha) \subset \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{N(j)} \mathcal{G}_n^j; \quad \sum_{j=1}^{\infty} \sum_{j=1}^{N(j)} h_\alpha(d(\mathcal{G}_n^j)) < \varepsilon$$

and $d(\mathcal{G}_n^j) \in S$ for $n = 1, 2, \ldots, N(j)$ and $j = 1, 2, \ldots$. This completes the proof of the lemma.

For $s = 3, 4, \ldots$ we put

$$b_s = \left(\frac{2s-1}{2s}\right)^{((2s-1)/2s)} \left(\frac{1}{2s}\right)^{(1/2s)}, \quad \alpha_s = \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{2s-1}{2s}$$

and observe, using Stirling's Formula, that $\alpha_s = O(s^{-1/2})$ as $s \to \infty$.

LEMMA 3. *Let $S = \{x_i\}$ be as in Lemma 1 and let $I_1, I_2, \ldots, I_t$ be closed intervals such that $\sum_{j=1}^{t} d(I_j) \leqslant 1$. For each $s \geqslant 3$, given any $\varepsilon > 0$, there is a* *sequence* $\{\mathcal{G}_n\}$ *of sets such that* $\mathcal{F}[\bigcup_{j=1}^t I_j] \subset \bigcup_{n=1}^\infty \mathcal{G}_n$, $\sum_{n=1}^\infty h_{\alpha_s}(d(\mathcal{G}_n)) < \varepsilon$ *and* $d(\mathcal{G}_n) \in S$ *for* $n = 1, 2, \ldots$.

PROOF. The proof of this lemma is in two sections.

(a) *Let* $I_1, I_2, \ldots, I_t$ *and* S *be as in the statement of the lemma. For each* $s \geqslant 3$, *given any* $\varepsilon > 0$, *there is a sequence* $\{\mathcal{G}_n\}$ *of sets such that* $\mathcal{F}[\bigcup_{j=1}^t I_j]\backslash\mathcal{F}((2s-1)/2s) \subset \bigcup_{n=1}^\infty \mathcal{G}_n$, $\sum_{n=1}^\infty h_{\alpha_s}(d(\mathcal{G}_n)) < \varepsilon$ *and* $d(\mathcal{G}_n) \in S$ *for* $n = 1, 2, \ldots$.

Now for each i, $\bigcup_{j=1}^t I_j$ can be covered by $N(i) = \sum_{j=1}^t \{[d(I_j)x_i^{-1}] + 1\}$ closed intervals each of length x_i. Let $I_1^i, I_2^i, \ldots, I_{N(i)}^i$ be an enumeration of the intervals. Then if $K \in \mathcal{F}[\bigcup_{j=1}^t I_j]\backslash\mathcal{F}((2s-1)/2s)$, cardinality $\{j: K \cap I_j^i \neq \varnothing\} \geqslant (2s-1)/2s$. Thus, for each i, $\mathcal{F}[\bigcup_{j=1}^t I_j]\backslash\mathcal{F}((2s-1)/2s)$ is contained in a set of $M = \sum \{\binom{N(i)}{k}: (2s-1)/2sx_i \leqslant k \leqslant N(i)\}$ spheres each of diameter x_i.

Now the number of terms in M is $O(x_i^{-1})$ and the binomial coefficients can be estimated by Stirling's Formula. We find, for any fixed $\eta > 0$, an estimate of the form

$$Mh_{\alpha_s}(x_i) = o((b_s^{-1}2^{-\alpha_s+\eta})^{x_i^{-1}}) \quad \text{as } i \to \infty.$$

So we need to prove that $2^{-\alpha_s} < b_s$ for all $s \geqslant 3$. This can easily be verified for $s = 3$. For $s \geqslant 4$ we use the inequality

$$(2s)^{1/2} > e > (1 + 1/(2s-1))^{2s-1}$$

to show that

$$\frac{2s-1}{2s}\log\frac{2s-1}{2s} > \frac{1}{4s}\log\frac{1}{2s}.$$

It is thus sufficient to prove that, for $s \geqslant 4$ we have

$$\alpha_s \log 2 + (3/4s)\log(1/2s) > 0.$$

We observe that since $2s > e^2 > (1 + 1/s)^{2s}$ we have

$$\frac{3}{2(2s+1)}\log\frac{1}{2s+2} > \frac{3}{4s}\log\frac{1}{2s}.$$

Using this fact we can easily deduce the required inequalities by induction on s and so (a) is proved.

(b) *Let* S *be as in the statement of the lemma and let* $I_1, I_2, \ldots, I_t$ *be closed intervals such that* $\sum_{j=1}^t d(I_j) < (2s-1)/2s$. *For each* $s \geqslant 3$, *given any* $\varepsilon > 0$, *there is a sequence* $\{\mathcal{G}_n\}$ *of sets such that* $\mathcal{F}[\bigcup_{j=1}^t I_j]\backslash\mathcal{F}(\alpha_s) \subset \bigcup_{n=1}^\infty \mathcal{G}_n$; $\sum_{n=1}^\infty h_{\alpha_s}(d(\mathcal{G}_n)) < \varepsilon$ *and* $d(\mathcal{G}_n) \in S$ *for* $n = 1, 2, \ldots$.

This is proved by induction on s. For $s = 3$ the statement is trivial since in that case $\mathfrak{F}[\bigcup_{j=1}^{t} I_j] \backslash \mathfrak{F}(\alpha_s) = \varnothing$. Now assume it is true for some $s \geqslant 3$ and let $I_1, I_2, \ldots, I_t$ be closed intervals with $\sum_{j=1}^{t} d(I_j) < (2s+1)/(2s+2)$. Then

$$\sum_{j=1}^{t} d\left(\frac{2s+2}{2s+1} I_j\right) < 1$$

and

$$\mathfrak{F}\left[\bigcup_{j=1}^{t} \frac{2s+2}{2s+1} I_j\right] \Big\backslash \mathfrak{F}(\alpha_s) \subset \left(\mathfrak{F}\left[\bigcup_{j=1}^{t} \frac{2s+2}{2s+1} I_j\right] \Big\backslash \mathfrak{F}\left(\frac{2s-1}{2s}\right)\right) \cup \left(\mathfrak{F}\left(\frac{2s-1}{2s}\right) \Big\backslash \mathfrak{F}(\alpha_s)\right).$$

But, by (a), there is a sequence $\{\mathcal{G}_n^0\}$ of sets such that

$$\mathfrak{F}\left[\bigcup_{j=1}^{t} \frac{2s+2}{2s+1} I_j\right] \Big\backslash \mathfrak{F}\left(\frac{2s-1}{2s}\right) \subset \bigcup_{n=1}^{\infty} \mathcal{G}_n^0; \quad \sum_{n=1}^{\infty} h_{\alpha_s}(d(\mathcal{G}_n^0)) < \frac{1}{2}\varepsilon$$

and $d(\mathcal{G}_n^0) \in (2s+2)S/(2s+1)$ for $n = 1, 2, \ldots$.

Let $\mathcal{I}_1, \mathcal{I}_2, \ldots$ be an enumeration of all the finite unions of closed intervals with rational endpoints whose total length is less than $(2s-1)/2s$. Then

$$\mathfrak{F}\left(\frac{2s-1}{2s}\right) \Big\backslash \mathfrak{F}(\alpha_s) \subset \bigcup_{i=1}^{\infty} (\mathfrak{F}[\mathcal{I}_i] \backslash \mathfrak{F}(\alpha_s)).$$

By induction hypothesis, for each i, there is a sequence $\{\mathcal{G}_n^i\}$ of sets such that

$$\mathfrak{F}[\mathcal{I}_i] \backslash \mathfrak{F}(\alpha_s) \subset \bigcup_{n=1}^{\infty} \mathcal{G}_n^i; \quad \sum_{n=1}^{\infty} h_{\alpha_s}(d(\mathcal{G}_n^i)) < \varepsilon 2^{-i-1}$$

and $d(\mathcal{G}_n^i) \in (2s+2)S/(2s+1)$ for $n = 1, 2, \ldots$. Hence

$$\mathfrak{F}\left[\bigcup_{j=1}^{t} I_j\right] \Big\backslash \mathfrak{F}(\alpha_{s+1}) \subset \bigcup_{i=0}^{\infty} \bigcup_{n=1}^{\infty} \frac{2s+1}{2s+2} \mathcal{G}_n^i;$$

$d((2s+1)\mathcal{G}_n^i/(2s+2)) = (2s+1)d(\mathcal{G}_n^i)/(2s+2) \in S$ for $n = 1, 2, \ldots$ and $i = 0, 1, 2, \ldots$; and

$$\sum_{i=0}^{\infty} \sum_{n=1}^{\infty} h_{\alpha_{s+1}}\left(\left(d \frac{2s+1}{2s+2} \mathcal{G}_n^i\right)\right) = \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} h_{\alpha_s}((d\mathcal{G}_n^i)) < \varepsilon.$$

This completes the induction and so (b) is proved.

We saw, in the proof, how (b) implies that given $\varepsilon > 0$ there is a sequence $\{\mathcal{G}_n\}$ of sets such that

$$\mathcal{F}\left(\frac{2s-1}{2s}\right)\Big\backslash\mathcal{F}(\alpha_s) \subset \bigcup_{n=1}^{\infty}\mathcal{G}_n; \quad \sum_{n=1}^{\infty} h_{\alpha_s}(d(\mathcal{G}_n)) < \varepsilon$$

and $d(\mathcal{G}_n) \in S$ for $n = 1, 2, \ldots$. Combining this fact with (a) and Lemma 2 we easily deduce Lemma 3.

LEMMA 4. *Let α and S be as in Lemma 1 and let $\varepsilon > 0$ be given. Then there are sets $\{\mathcal{G}_n\}$ such that $\mathcal{F}[0,1] \subset \bigcup_{n=1}^{\infty}\mathcal{G}_n$; $\sum_{n=1}^{\infty} h_\alpha(d(\mathcal{G}_n)) < \varepsilon$ and $d(\mathcal{G}_n) \in S$ for $n = 1, 2, \ldots$.*

PROOF. Choose $s \geqslant 3$ so that $0 < \alpha_s < \alpha$. By Lemma 3 there are sets $\{\mathcal{G}_n\}$ such that $\mathcal{F}[0,1] \subset \bigcup_{n=1}^{\infty}\mathcal{G}_n$; $\sum_{n=1}^{\infty} h_{\alpha_s}(d(\mathcal{G}_n)) < \varepsilon$ and $d(\mathcal{G}_n) \in S$ for $n = 1, 2, \ldots$. Thus

$$\sum_{n=1}^{\infty} h_\alpha(d(\mathcal{G}_n)) < \sum_{n=1}^{\infty} h_{\alpha_s}(d(\mathcal{G}_n)) < \varepsilon$$

as required.

PROOF OF THEOREM 1. Let $A = \liminf_{x\to 0}\{-(x \log h(x))^{-1}\}$, then $0 \leqslant A < \infty$. Choose $\alpha \in (0,1)$ so that $A \log 2 < \alpha^{-1}$. Then there is a sequence $S = \{x_i\}$ of positive real numbers with $\lim_{i\to\infty} x_i = 0$ and $-(x_i \log h(x_i))^{-1} < (\alpha \log 2)^{-1}$ for $i = 1, 2, \ldots$. Hence $h(x_i) < h_\alpha(x_i)$ for $i = 1, 2, \ldots$. Thus, by Lemma 4, given any $\varepsilon > 0$ there are sets $\{\mathcal{G}_n\}$ such that $\mathcal{F}[0,1] \subset \bigcup_{n=1}^{\infty}\mathcal{G}_n$; $\sum_{n=1}^{\infty} h_\alpha(d(\mathcal{G}_n)) < \varepsilon$ and $d(\mathcal{G}_n) \in S$ for $n = 1, 2, \ldots$. Hence

$$\sum_{n=1}^{\infty} h(d(\mathcal{G}_n)) < \sum_{n=1}^{\infty} h_\alpha(d(\mathcal{G}_n)) < \varepsilon$$

and so $h - m(\mathcal{F}[0,1]) = 0$ as required.

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